# Some Examples of Nonsingular Projective Varieties with Semiample Tangent Bundle I

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If $X$ is a nonsingular complete intersection in a projective space $\mathbf{P}^n$ with $n > \dim X \geqq 2$ and is not contained in any hyperplane, the tangent bundle of $X$ is not semiample unless $X$ is a quadric hypersurface.

## Introduction

In this series of notes, we shall study some examples of nonsingular projective varieties with semiample tangent bundle.

In 1979, Mori [5] proved the so-called Hartshorne conjecture: Every nonsingular projective variety with ample tangent bundle is a projective space. Inspired by this result, Campana and Peternell [1] classified nonsingular projective varieties with numerically effective tangent bundle, in case of dimension 3. On the other hand, in [3] we have studied nonsingular projective varieties with semiample tangent bundle. Our assumption on tangent bundle is rather stronger than Campana and Peternell's one. It seems that varieties with semiample tangent bundle are of very special type. Therefore we might hope to find out all of these varieties, and might hope that they would play a very important role in the classification theory of algebraic varieties.

In the present note, we shall investigate nonsingular complete intersections in complex projective spaces $\mathbf{P}^n$ and show the following:

**Theorem.** *Let $X$ be a nonsingular complete intersection in a projective space $\mathbf{P}^n$ with $n > \dim X \geqq 2$, and assume that $X$ is not contained in any hyperplane. Then the tangent bundle of $X$ is semiample if and only if $X$ is a quadric hypersurface.*

## Proof of Theorem

Given a vector bundle $\mathscr{E}$, we denote by $\mathbf{P}(\mathscr{E})$ the projective space bundle associated with $\mathscr{E}$ and by $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ the tautological line bundle of $\mathbf{P}(\mathscr{E})$. A vector bundle $\mathscr{E}$ is said to be semiample if there exists a positive integer $k$ such that $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(k) = \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)^{\otimes k}$ is spanned by its global sections.

Let $X$ be a nonsingular subvariety in $\mathbf{P}^n$. Then we denote by $\mathscr{T}_X$, $\omega_X$ the tangent bundle, the canonical line bundle of $X$ respectively. We put $\mathscr{O}_X(k) = \mathscr{O}_{\mathbf{P}^n}(k) \otimes \mathscr{O}_X$.

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**Proposition 1.** *Let $X$ be a nonsingular subvariety in $\mathbf{P}^n$. Then the vector bundle $\mathscr{T}_X \otimes \omega_X \otimes \mathscr{O}_X(\dim X)$ is semiample.*

*Proof.* Let $\mathscr{N}$ be the normal bundle of $X$. Then we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathscr{O}_X \arrow[r, equal] \arrow[d] & \mathscr{O}_X \arrow[d] & & \\
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \oplus^{n+1}\mathscr{O}_X(1) \arrow[r] \arrow[d] & \mathscr{N} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{T}_X \arrow[r] \arrow[d] & \mathscr{T}_{\mathbf{P}^n} \otimes \mathscr{O}_X \arrow[r] \arrow[d] & \mathscr{N} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathscr{F}$ is the kernel of the natural surjective homomorphism $\oplus^{n+1}\mathscr{O}_X(1) \to \mathscr{N}$. Note that $\mathscr{F}$ is a vector bundle such that rank$\mathscr{F} = \dim X + 1$. Let $\mathscr{F}^*$ be the dual bundle of $\mathscr{F}$. Then $\mathscr{F}^* \otimes \mathscr{O}_X(1)$ is spanned by its global sections and hence semiample. On the other hand, from the above diagram we obtain $\det \mathscr{F}^* \cong \omega_X$. Then by Corollary 1 in [2], we find that the vector bundle $\mathscr{F} \otimes \omega_X \otimes \mathscr{O}_X(\dim X)$ is semiample. Since the tangent bundle $\mathscr{T}_X$ is a quotient of $\mathscr{F}$, the vector bundle $\mathscr{T}_X \otimes \omega_X \otimes \mathscr{O}_X(\dim X)$ is also semiample. Q.E.D.

**Corollary 2.** *Let $X$ be a nonsingular quadric hypersurface. Then the tangent bundle $\mathscr{T}_X$ of $X$ is semiample.*

*Proof.* Since $X$ is a hypersurface of degree 2, $\omega_X \cong \mathscr{O}_X(-\dim X)$. Therefore the result follows immediately from Proposition 1. Q.E.D.

Thus we have proved the if part of Theorem.

Given a vector bundle $\mathscr{E}$ on a nonsingular projective variety, we denote by $c_i(\mathscr{E})$ (resp. $s_i(\mathscr{E})$) the $i$-th Chern (resp. Segre) class of $\mathscr{E}$. Put $c(\mathscr{E}) = \sum c_i(\mathscr{E})$, $s(\mathscr{E}) = \sum s_i(\mathscr{E})$. Then we have $s(\mathscr{E})c(\mathscr{E}) = 1$.

**Proposition 3.** *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular projective variety $X$ of dimension $m$. Then*

$$s_i(\mathscr{E}^*).H^{m-i} \geqq 0$$

*for every* $i \geqq 1$ *and every ample divisor class* $H$ *on* $X$. *Furthermore if* $s_i(\mathscr{E}^*).H_0^{m-i} = 0$ *for some* $i$ *and some ample divisor class* $H_0$, *then*

$$s_j(\mathscr{E}^*).H^{m-j} = 0$$

*for every* $j \geqq i$ *and for every ample divisor class* $H$.

*Proof.* Since $\mathscr{E}$ is semiample, there exists a positive integer $k$ for which the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(k)$ is spanned by its global sections. Then for every $i \geqq 1$ the class $k^i c_1(\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1))^i$ is represented by a nonnegative cycle, and if $c_1(\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1))^i = 0$ for some $i$, then $c_1(\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1))^j = 0$ for every $j \geqq i$. Hence the result follows from Lemma 1.8 in [4]. Q.E.D.

To prove the only if part of Theorem, we let $X \subset \mathbf{P}^n$ be a nonsingular complete intersection of hypersurfaces $H_l$ of degree $d_l$ $(l = 1, 2, \cdots, n - \dim X)$. Since $X$ is not contained in any hyperplane, $d_l \geqq 2$ for every $l$. Set $m = \dim X$ and set $r = n - m$. If $r = 1$ and $d_1 = 2$, then $X$ is a quadric hypersurface. Therefore in what follows we assume that $r \geqq 2$ or $d_1 \geqq 3$. Let $\Omega_X^1$ (resp. $\Omega_{\mathbf{P}^n}^1$) be the cotangent bundle of $X$ (resp. $\mathbf{P}^n$). Then by Proposition 3 we have only to show that $s_i(\Omega_X^1).H^{m-i} < 0$ for some $1 \leqq i \leqq m$. From the exact sequence $0 \to \mathscr{N}^* \to \Omega_{\mathbf{P}^n}^1 \otimes \mathscr{O}_X \to \Omega_X^1 \to 0$, we obtain

$$s(\Omega_X^1) = c(\mathscr{N}^*) s(\Omega_{\mathbf{P}^n}^1 \otimes \mathscr{O}_X).$$

On the other hand, we have

$$\begin{aligned} c(\mathscr{N}^*) &= (1 - d_1 H)(1 - d_2 H) \cdots (1 - d_r H), \\ s(\Omega_{\mathbf{P}^n}^1 \otimes \mathscr{O}_X) &= s(\mathscr{O}_X(-1))^{n+1} \\ &= (1 + H + H^2 + \cdots + H^m)^{n+1}, \end{aligned}$$

where $H = c_1(\mathscr{O}_X(1))$. Thus we obtain

$$s(\Omega_X^1) = (1 - d_1 H)(1 - d_2 H) \cdots (1 - d_r H)(1 + H + H^2 + \cdots + H^m)^{n+1}.$$

Therefore it remains to prove the following:

**Lemma 4.** *Let* $m, r, d_1, d_2, \cdots, d_r$ *be positive integers and assume that* $m \geqq 2$, $d_1 \geqq d_2 \geqq \cdots \geqq d_r \geqq 2$. *Put*

$$(1 - d_1 x)(1 - d_2 x) \cdots (1 - d_r x)(1 + x + x^2 + \cdots + x^m)^{m+r+1} = \sum a_i x^i.$$

*Then* $a_i < 0$ *for some* $1 \leqq i \leqq m$ *unless* $r = 1$ *and* $d_1 = 2$.

*Proof.* For each $1 \leqq j \leqq r$, put

$$(1 - d_1 x)(1 - d_2 x) \cdots (1 - d_j x)(1 + x + x^2 + \cdots + x^m)^{m+r+1} = \sum a_{i,j} x^i.$$

Then $a_i = a_{i,r}$ for every $i$. We have

$$a_{i,j} = \sum_{0 \leqq k \leqq \min\{i,j\}} (-1)^k S_{j,k} \cdot \frac{(m+r+i-k)!}{(i-k)!(m+r)!},$$

where $S_{j,k}$ is the $k$-th elementary symmetric function of $d_1, d_2, \cdots, d_j$. In particular,

$$\begin{aligned} a_{1,r} &= (m+r+1) - (d_1 + d_2 + \cdots + d_r) \\ &= m + 1 - \{(d_1 - 1) + (d_2 - 1) + \cdots + (d_r - 1)\}, \\ a_{m,1} &= \frac{(2m+r)!}{m!(m+r)!} - d_1 \cdot \frac{(2m+r-1)!}{(m-1)!(m+r)!} \\ &= \{r - m(d_1 - 2)\} \cdot \frac{(2m+r-1)!}{m!(m+r)!}. \end{aligned}$$

If $r = 1$, then $d_1 \geqq 3$. Therefore we can immediately see that $a_m = a_{m,1} < 0$.

In what follows, assume that $r \geqq 2$. If $d_1 \geqq 3$ and if $a_1 \geqq 0$, then $m \geqq r$. This implies that $a_{m,1} \leqq 0$. If $a_{i,j} \leqq 0$ for some $1 \leqq i \leqq m$ and $1 \leqq j \leqq r-1$, then there exists an integer $1 \leqq I \leqq i$ such that $a_{I,j} \leqq 0$ and $a_{k,j} > 0$ for every $0 \leqq k < I$, and then it follows that $a_{I,j+1} < 0$. Hence by induction we find that $a_i = a_{i,r} < 0$ for some $1 \leqq i \leqq m$.

Finally we assume that $d_1 = d_2 = \cdots = d_r = 2$. In this case, we have

$$\begin{aligned} \sum a_i x^i &= (1-2x)^r (1 + x + x^2 + \cdots + x^m)^{m+r+1} \\ &= \{f(x)\}^{r-2} (1-2x)^2 (1 + x + x^2 + \cdots + x^m)^{m+3}, \end{aligned}$$

where

$$\begin{aligned} f(x) &= (1-2x)(1 + x + x^2 + \cdots + x^m) \\ &= 1 - x - x^2 - \cdots - x^m - 2x^{m+1}. \end{aligned}$$

Put

$$(1-2x)^2 (1 + x + x^2 + \cdots + x^m)^{m+3} = \sum b_i x^i.$$

Then

$$\begin{aligned} b_m &= \frac{(2m+2)!}{m!(m+2)!} - 4 \cdot \frac{(2m+1)!}{(m-1)!(m+2)!} + 4 \cdot \frac{(2m)!}{(m-2)!(m+2)!} \\ &= (2-2m) \cdot \frac{(2m)!}{m!(m+2)!} \\ &< 0. \end{aligned}$$

Hence in the same manner as above, we can show that $a_i < 0$ for some $1 \leqq i \leqq m$.

Q.E.D.

## References

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